IBM: parameter symmetry, hidden symmetries and transformations of boson operators*

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Abstract

A symmetry of the parameter space of interacting boson models IBM-1 and IBM-2 is studied. The symmetry is associated with linear canonical transformations of boson operators, or, equivalently, with the existence of different realizations of the symmetry algebras of the models. The relevance of the parameter symmetry to physical observables is discussed.

1 Introduction

It has been established recently [1, 2, 3] that the Hamiltonian of the simplest version of the interacting boson model IBM-1 [4] possesses an additional symmetry, the so-called parameter symmetry, that is a symmetry of the parameter space of the model. The symmetry manifests itself in the existence of two sets of the Hamiltonian parameters that generate identical spectra.

The IBM-1 Hamiltonian has an algebraic structure characterized by the U(6) algebra. The spectrum and the eigenfunctions can be found analytically in three particular cases (the U(5), SU(3) and SO(6) dynamical symmetry (DS) limits). A non-trivial issue of the parameter symmetry is that it establishes an equivalence between the exactly solvable IBM-1 DS limits and transitional IBM-1 Hamiltonians of a general form.

The parameter symmetry is associated with canonical transformations of boson operators linking different realizations of SU(3) and SO(6) subalgebras in the U(6) algebra [2, 3].

In this paper we propose a generalization of the parameter symmetry concept on the case of IBM-2, the proton-neutron version of IBM. After a survey of the parameter symmetry of IBM-1, we turn the discussion to the structure of the general IBM-2 Hamiltonian and derive the IBM-2 parameter symmetry relations.

2 Parameter Symmetry of IBM-1

Within IBM-1 nuclear states are labelled by a fixed total number N of bosons of two types, s and d, with quantum numbers $l^\pi = 0^+$ and $l^\pi = 2^+$, respectively [4]. The U(6) algebra is generated by 36 bilinear combinations of boson operators: s^+s, $d^+_\mu d_\nu$, $d^+_\mu s$, $s^+ d_\mu$. The general IBM-1 Hamiltonian can be expressed as [4]

$$\begin{aligned} H(\{k_i\}) \;=\; & H_0 + k_1 C_1[\mathrm{U}(5)] + k_2 C_2[\mathrm{U}(5)] + k_3 C_2[\mathrm{SO}(5)] \\ & + k_4 C_2[\mathrm{SO}(3)] + k_5 C_2[\mathrm{SO}(6)] + k_6 C_2[\mathrm{SU}(3)]\,, \end{aligned} \tag{1}$$

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Table 1: Generators of U(6), U(5), SU(3), SO(6), SO(5) and SO(3) algebras

Algebra	Generators
U(6)	$[d^+\times s]^{(2)}_\mu$, $[s^+\times\tilde{d}]^{(2)}_\mu$, $S=[s^+\times s]^{(0)}_0$, $T^\lambda_\mu=[d^+\times\tilde{d}]^{(\lambda)}_\mu$, $\lambda=0,1,2,3,4$
U(5)	$T^\lambda_\mu=[d^+\times\tilde{d}]^{(\lambda)}_\mu$, $\lambda=0,1,2,3,4$
SU(3)	$Q_\mu=[d^+\times s+s^+\times\tilde{d}]^{(2)}_\mu-\frac{\sqrt{7}}{2}[d^+\times\tilde{d}]^{(2)}_\mu$, $T^1_\mu=[d^+\times\tilde{d}]^{(1)}_\mu$
SO(6)	$Q^0_\mu=[d^+\times s+s^+\times\tilde{d}]^{(2)}_\mu$, $T^\lambda_\mu=[d^+\times\tilde{d}]^{(\lambda)}_\mu$, $\lambda=1,3$
SO(5)	$T^\lambda_\mu=[d^+\times\tilde{d}]^{(\lambda)}_\mu$, $\lambda=1,3$
SO(3)	$T^1_\mu=[d^+\times\tilde{d}]^{(1)}_\mu$

where C_1 and C_2 stand for the first and the second rank Casimir invariants of the algebras entering the reduction chains of the U(6) algebra:

$$\mathrm{U}(6)\ \begin{array}{ll}\nearrow & \mathrm{U}(5)\supset\mathrm{SO}(5)\supset\mathrm{SO}(3) \qquad \mathrm{I}\\ \rightarrow & \mathrm{SU}(3)\supset\mathrm{SO}(3) \qquad \mathrm{II}\\ \searrow & \mathrm{SO}(6)\supset\mathrm{SO}(5)\supset\mathrm{SO}(3) \qquad \mathrm{III}\end{array} \tag{2}$$

We define the Casimir operators as in the book [5]:

$$\begin{array}{ll} C_1[\mathrm{U}(5)]=n_d\ , & C_2[\mathrm{SO}(5)]=2(T^3\cdot T^3)+2(T^1\cdot T^1)\ ,\\ C_2[\mathrm{U}(5)]=n_d(n_d{+}4)\ , & C_2[\mathrm{SO}(6)]=N(N{+}4)-4\,(P^+\cdot P)\ ,\\ C_2[\mathrm{SO}(3)]=10(T^1\cdot T^1)\ , & C_2[\mathrm{SU}(3)]=2(Q\cdot Q)+\frac{15}{2}(T^1\cdot T^1)\ ,\end{array} \tag{3}$$

where generators of the groups entering reduction chains (2) are given in Table 1,

$$n_d=(d^+\cdot\tilde{d})\ ,\quad P=\tfrac{1}{2}\left((\tilde{d}\cdot\tilde{d})-(s\cdot s)\right)\ , \tag{4}$$

$\tilde{d}_\mu=(-1)^\mu d_{-\mu}$, $(t\cdot u)$ and $[t\times u]^{(\lambda)}_\mu$ are scalar and tensor products, respectively, of spherical tensors t and u.

Dynamical symmetry limits correspond to the cases when the Hamiltonian involves Casimir operators belonging to one of the reduction chains (2) only, and hence the eigenvalues and eigenfunctions can be found analytically. The spectrum of the IBM Hamiltonian in the case of a DS limit is one of the typical nuclear spectra [4]: vibrational in the U(5) DS limit ($k_5{=}k_6{=}0$), rotational in the SU(3) DS limit ($k_1{=}k_2{=}k_3{=}k_5{=}0$) and γ-unstable in the SO(6) DS limit ($k_1{=}k_2{=}k_6{=}0$). A transitional nuclear Hamiltonian that does not possess any DS, is conventionally believed to generate a spectrum different from those corresponding to any of the DS limits.

As we have shown in Refs. [1, 2, 3], the IBM-1 Hamiltonian possesses a parameter symmetry, namely:

Hamiltonians $H(\{k_i\})$ and $H(\{k'_i\})$ defined by Eq. (1) *have identical spectra of eigenvalues if the corresponding parameter sets $\{k_i\}$ and $\{k'_i\}$ are related as*

$$\begin{array}{l} H'_0=H_0,\ k'_1=k_1+2k_6,\ k'_2=k_2+2k_6,\ k'_3=k_3-6k_6,\\ k'_4=k_4+2k_6,\ k'_5=k_5+4k_6,\ k'_6=-k_6\end{array} \tag{5}$$

in the case $k_6\neq 0$, or as

$$\begin{array}{l} H'_0=H_0+10k_5N,\ k'_1=k_1+4k_5(N{+}2),\ k'_2=k_2-4k_5,\\ k'_3=k_3+2k_5,\ k'_4=k_4,\ k'_5=-k_5,\ k'_6=0\end{array} \tag{6}$$

in the case $k_6 = 0$.

This statement was proved (see Refs. [1, 2, 3] for details) by constructing a unitary transformation U such that $H(\{k'_i\}) = UH(\{k_i\})U^{-1}$.

Thus, for any set of the IBM-1 parameters there is another set which generates the identical spectrum. The only exception is the U(5) DS limit when the two sets of the parameters coincide as is seen from (6).

One of the most intriguing issues of the parameter symmetry is that it establishes the equivalence of the nuclear spectrum corresponding to a certain DS to the spectrum of a transitional IBM-1 Hamiltonian. As follows from Eqs. (5), the rotational spectrum of the SU(3) DS limit (k_1=k_2=k_3=k_5=0) appears to be equivalent to the spectrum of the transitional Hamiltonian with the set of parameters $\{k'_i\} \equiv \{k'_1$=$2k_6$, k'_2=$2k_6$, k'_3=$-6k_6$, k'_4=k_4+$2k_6$, k'_5=$2k_6$, k'_6=$-k_6\}$ that does not correspond to any DS. Similarly, it follows from (6), that the γ-unstable spectrum of the SO(6) DS limit (k_1=k_2=k_6=0) can be obtained with the set of parameters $\{k'_i\} \equiv \{k'_6$=0, k'_1=$8(N$+2$)k_5$, k'_2=$-8k_5$, k'_3=k_3+$4k_5$, k'_4=k_4, k'_5=$-k_5\}$ corresponding to the U(5)–SO(6) transitional nuclear spectrum. In Ref. [6], such transitional Hamiltonians were referred to as the ones possessing a hidden symmetry.

To reveal the origin of the parameter symmetry, we note that there is an ambiguity in definition of boson operators within IBM [3, 4, 5, 7, 8]. One can apply to the boson operators gauge transformations $R_s(\varphi_s)$ and $R_d(\varphi_d)$ defined as [7, 4]

$$\begin{array}{ll} R_s(\varphi_s)\, s^+ = \exp(\mathrm{i}\varphi_s/2)\, s^+ \,, & R_s(\varphi_s)\, s = \exp(-\mathrm{i}\varphi_s/2)\, s \,, \\ R_d(\varphi_d)\, d^+_\mu = \exp(\mathrm{i}\varphi_d/2)\, d^+_\mu \,, & R_d(\varphi_d)\, \tilde{d}_\mu = \exp(-\mathrm{i}\varphi_d/2)\, \tilde{d}_\mu \,. \end{array} \tag{7}$$

Note that these transformations are the canonical ones, i.e. they do not violate the boson commutation relations. However, the structure of the IBM Hamiltonian implies severe restrictions on the use of the transformations (7). For example, in the case of the general IBM Hamiltonian, one can only apply to the Hamiltonian the gauge transformation $R(\varphi) \equiv R_s(\varphi_s) \times R_d(\varphi_d)$ with $\varphi \equiv (\varphi_s - \varphi_d)/2 = 0,\, \pi$ and arbitrary $\tilde{\varphi} \equiv (\varphi_s + \varphi_d)/2$ [7, 8]. Similarly, in the case of the transitional SO(6)–U(5) IBM Hamiltonian with $k_6 = 0$, one can use the gauge transformation $R(\varphi)$ with $\varphi = 0,\, \dfrac{\pi}{2},\, \pi,\, \dfrac{3\pi}{2}$. One can also apply to the boson operators the particle-hole conjugation $\tilde{R}$ [9, 5, 8] defined as

$$\begin{array}{ll} \tilde{R}\, s^+ = s \,, & \tilde{R}\, s = -s^+ \,, \\ \tilde{R}\, d^+_\mu = \tilde{d}_\mu \,, & \tilde{R}\, \tilde{d}_\mu = -d^+_\mu \,, \end{array} \tag{8}$$

and operators $\tilde{R}(\varphi) \equiv \tilde{R} \times R(\varphi)$ that are consistent with the Hamiltonian structure provided that $\varphi = 0,\, \dfrac{\pi}{2},\, \pi,\, \dfrac{3\pi}{2}$ in the case $k_6 = 0$ or $\varphi = 0,\, \pi$ in the case $k_6 \neq 0$. The operators $R(\varphi)$ and $\tilde{R}(\varphi)$ comprise a point group studied elsewhere [8].

We use the following notations for operators subjected to the transformations $R(\varphi)$ and $\tilde{R}(\varphi)$: ${}^{\alpha}O \equiv R(\varphi)\, O$ and ${}^{-\alpha}O \equiv \tilde{R}(\varphi)\, O$, $\alpha = \varphi/\pi$.

Hamiltonians ${}^{\alpha}H(\{k_i\}) \equiv R(\varphi)\, H(\{k_i\})$ are, of course, isospectral with the initial Hamiltonian $H(\{k_i\})$ [Hamiltonians ${}^{-\alpha}H(\{k_i\}) \equiv \tilde{R}(\varphi)\, H(\{k_i\})$ may be not isospectral with $H(\{k_i\})$; however, one can always find a simple and not very restrictive constraint on the parameters k_i that will guarantee the isospectrality of ${}^{-\alpha}H(\{k_i\})$ and $H(\{k_i\})$]. Thus we can use transformations (7) [and in some cases (8)] to study parameter symmetries and hidden symmetries of IBM.

For example, the Hamiltonian ${}^{1}H(\{k_i\}) \equiv R(\pi)\, H(\{k_i\})$ is isospectral but not equivalent to the initial Hamiltonian $H(\{k_i\})$. Using Table 1 and expressions (1), (3), (4) and (7), one can obtain [3] that ${}^{1}H(\{k_i\}) = H(\{k'_i\})$ where the set of parameters $\{k'_i\}$ is defined by the parameter symmetry relation (5). Thus, the transformation $R(\pi)$ is equivalent to the parameter symmetry transformation (5).

Note that, as is seen from Eqs. (3)–(4) and Table 1, only the Casimir operator C_2[SU(3)] and the SU(3) generator Q_μ are changed under the transformation $R(\pi)$:

$$R(\pi)\, Q_\mu = {}^{1}Q_\mu = -[d^+ \times s \,+\, s^+ \times \tilde{d}]^{(2)}_\mu - \frac{\sqrt{7}}{2}[d^+ \times \tilde{d}]^{(2)}_\mu \,. \tag{9}$$

The quadrupole operators Q_μ and ${}^1Q_\mu$ correspond to different embeddings of the SU(3) subalgebra in the U(6) algebra [see also [9] for other realizations of SU(3)]. Using parameter symmetry transformation (5) it is easy to express the Casimir operator $C_2\big[\mathrm{SU}^1(3)\big]$ of the $\mathrm{SU}^1(3)$ algebra associated with the quadrupole operator (9) through $C_2\left[\mathrm{SU}(3)\right]$ and Casimir operators of other algebras [2, 3]:

$$\begin{aligned} C_2\big[\mathrm{SU}^1(3)\big] = \quad & 2C_1[\mathrm{U}(5)] + 2C_2[\mathrm{U}(5)] - 6C_2[\mathrm{SO}(5)] \\ & + 2C_2[\mathrm{SO}(3)] + 4C_2[\mathrm{SO}(6)] - C_2[\mathrm{SU}(3)]\,. \end{aligned} \tag{10}$$

In the case $k_6 = 0$ we have ${}^1H(\{k_i\}) = H(\{k_i\})$ and the transformation $R(\pi)$ does not generate the parameter symmetry. However, in this case we can apply the transformation $R(\pi/2)$ to the Hamiltonian. Using Table 1 and expressions (1), (3), (4) and (7), we obtain [3] that ${}^{1/2}H(\{k_i\}) = H(\{k'_i\})$ where the set of parameters $\{k'_i\}$ is defined by the parameter symmetry relation (6). Hence, the transformation $R(\pi/2)$ is equivalent to the parameter symmetry transformation (6).

With the help of the transformation $R(\pi/2)$ we obtain a new monopole operator:

$$\overline{P} \equiv \quad {}^{1/2}P = R(\pi/2)\,P = \frac{1}{2}\left((\tilde{d}\cdot\tilde{d}) + (s\cdot s)\right). \tag{11}$$

This monopole operator corresponds to an alternative embedding of the SO(6) subalgebra in the U(6) algebra [7, 5]. Using the parameter symmetry relation (6) it is easy to obtain [2, 3] the following expression for the Casimir operator of the $\overline{\mathrm{SO}(6)}$ algebra associated with the monopole operator $\overline{P}$:

$$C_2\left[\overline{\mathrm{SO}(6)}\right] = 10N{+}4(N{+}2)\,C_1[\mathrm{U}(5)]{-}4C_2[\mathrm{U}(5)]{+}2C_2[\mathrm{SO}(5)]{-}C_2[\mathrm{SO}(6)]\,. \tag{12}$$

Note, that the Casimir operators of alternatively embedded algebras $\mathrm{SU}^1(3)$ and $\overline{\mathrm{SO}}(6)$ are not independent from the Casimir operators of other algebras and should not be included into the general Hamiltonian (1).

The transformations $\tilde{R}(\varphi)$ do not generate new parameter symmetries. However one more parameter symmetry relation can be obtained in the case of IBM-1 that is not associated with the transformations $R(\varphi)$ and $\tilde{R}(\varphi)$ (see [3] for a more detailed discussion).

Usually in applications the Hamiltonian parameters $\{k_i\}$ are obtained by the fit to nuclear spectra. Due to the parameter symmetry, the fit of the parameters appears to be ambiguous. To discriminate between the two sets of parameters giving rise to identical spectra, it is natural to study electromagnetic transitions.

In the consistent-Q formalism (CQF) [10], both monopole-monopole $(P^+\cdot P)$ and quadrupole-quadrupole $(Q\cdot Q)$ interactions are replaced in the Hamiltonian by a single term $(Q^\chi\cdot Q^\chi)$ where the generalized quadrupole operator

$$Q^\chi_\mu = [d^+\times s + s^+\times\tilde{d}]^{(2)}_\mu + \chi\,[d^+\times\tilde{d}]^{(2)}_\mu. \tag{13}$$

Operator Q^χ is used for calculations of $E2$-transition rates within CQF. Applying transformation $R(\pi)$ to the Hamiltonian $H(\{k_i\})$, we find out that the only term in the new Hamiltonian ${}^1H(\{k_i\}) = R(\pi)\,H(\{k_i\})$ that differs from the corresponding term in the initial Hamiltonian $H(\{k'_i\})$, is the generalized quadrupole-quadrupole interaction:

$$\left({}^1Q^\chi\cdot{}^1Q^\chi\right) = R(\pi)\,\left(Q^\chi\cdot Q^\chi\right), \tag{14}$$

where

$${}^1Q^\chi_\mu \equiv R(\pi)\,Q^\chi_\mu = -Q^{-\chi}_\mu\,. \tag{15}$$

The consistent transformation of the $E2$ transition operator (13) according to (15) and of the generalized quadrupole-quadrupole interaction in the Hamiltonian according to (14), guarantees that the $E2$ transition rates remain unchanged. Therefore in the general case $\chi \neq 0$ that corresponds to $k_6 \neq 0$, the $E2$ transition rates cannot be used to distinguish between two sets of Hamiltonian parameters $\{k_i\}$ and $\{k'_i\}$ related by the parameter symmetry (5), at least within the CQF formalism. if it is believed that

the CQF ansatz is an adequate prescription for the electromagnetic transition operator. We note that in the general case $\chi \neq 0$ the type of the generalized quadrupole-quadrupole interaction [whether it is of the form $(Q^\chi \cdot Q^\chi)$ or $({}^1Q^\chi \cdot {}^1Q^\chi)$] is unambiguously determined by the set of the Hamiltonian parameters $\{k_i\}$. We have shown in [3] that this is due to the fact that the generalized quadrupole-quadrupole interaction includes the monopole-monopole term $(P^+ \cdot P)$.

In the case $\chi = 0$ ($k_6 = 0$), we apply the transformation $R(\pi/2)$ to the Hamiltonian and to the quadrupole operator Q^0_μ to obtain

$$\overline{Q^0_\mu} \equiv \ {}^1\!/_2 Q^0_\mu \equiv R(\pi/2)\, Q^0_\mu = -\mathrm{i}[d^+ \times s \ - \ s^+ \times \tilde{d}]^{(2)}_\mu \,. \tag{16}$$

However, in this case the generalized quadrupole-quadrupole interaction is ambiguous. As we have shown in Ref. [3], the parameter symmetry relation (6) can be used to derive

$$\left(Q^0 \cdot Q^0\right) = -\left(\overline{Q^0} \cdot \overline{Q^0}\right) + 10N + 4(N+2)\, C_1[\mathrm{U}(5)] - 4C_2[\mathrm{U}(5)] - 2C_2[\mathrm{SO}(5)] \,. \tag{17}$$

Thus in the case $k_6 = 0$ the IBM-1 Hamiltonian can be expressed either through $\left(Q^0 \cdot Q^0\right)$ or alternatively through $\left(\overline{Q^0} \cdot \overline{Q^0}\right)$. As a result, the definition of the $E2$ transition operator appears to be ambiguous. Due to this ambiguity, the electromagnetic transition rates cannot be used to distinguish among two sets of Hamiltonian parameters $\{k_i\}$ and $\{k'_i\}$ related by the parameter symmetry (6). The origin of the ambiguity of the generalized quadrupole-quadrupole interaction is that the quadrupole-quadrupole interaction $(Q \cdot Q)$ is not present in the Hamiltonian in the case $k_6 = 0$ and the operators $\left(Q^0 \cdot Q^0\right)$ and $\left(\overline{Q^0} \cdot \overline{Q^0}\right)$ within CQF replace the monopole-monopole term $(P^+ \cdot P)$ in the Hamiltonian (see Ref. [3] for more details).

There is another possibility to distinguish among the two Hamiltonians related by the parameter symmetry in the case $k_6 = 0$. This possibility stems from the N-dependence of the parameter symmetry (6). Since the relations (6) involve the total number of bosons N, the two sets of the parameters can generate identical spectra for some particular nucleus only, the predictions for the spectra of its isotopes or isotones should differ. It is conventionally supposed (see for example Ref. [11]) that the spectra of neighboring even-even nuclei are described by the same set of the IBM parameters, hence one can discriminate between the parameter sets $\{k_i\}$ and $\{k'_i\}$ related according to (6) by comparing the spectra of the neighboring nuclei.

This is illustrated by Fig. 1 where the spectra of three Pt isotopes are presented. The set of parameters $k_1 = k_2 = k_6 = 0$, $k_3 = 50$ keV, $k_4 = 10$ keV, $k_5 = -42.75$ keV was suggested in [4] for the description of ^{196}Pt (N=6) within the SO(6) DS limit of IBM. The corresponding spectra are given in the left columns labelled by SO(6). The set of parameters $k'_1 = -1368$ keV, $k'_2 = 171$ keV, $k'_3 = -35.5$ keV, $k'_4 = 10$ keV, $k'_5 = 42.75$ keV and $k'_6 = 0$ is obtained using (6) with $N = 6$. The corresponding spectra are given in the right columns labelled by PS. The SO(6) and parameter symmetry spectra are, of course, identical in the case of ^{196}Pt but differ for other Pt isotopes.

As is seen from (1), (3) and (4), the transformation $R(\pi/2)$ or, equivalently, the parameter symmetry transformation (6), changes the sign of the monopole-monopole interaction $(P^+ \cdot P)$ in the Hamiltonian. This sign change manifests itself in the spectra of neighboring nuclei. It is usually supposed that the pairing (monopole-monopole) interaction should be attractive, i.e. $k_5 < 0$. Note that the set of parameters suggested in Ref. [4] with attractive pairing interaction fitted to ^{196}Pt, reproduces the experimental data on ^{192}Pt and ^{194}Pt better (see Fig. 1) than the other set with $k'_5 > 0$.

The generalized quadrupole-quadrupole interaction $(Q^\chi \cdot Q^\chi)$ incorporates both quadrupole-quadrupole $(Q \cdot Q)$ and pairing $(P^+ \cdot P)$ interactions. The transformation $R(\pi)$, or, equivalently, the parameter symmetry transformation (5) changes only the sign of the quadrupole-quadrupole interaction $(Q \cdot Q)$ in the Hamiltonian, as is seen from Table 1 and Eqs. (1), (3) and (4); the monopole-monopole and other multipole-multipole terms are not effected by the transformation $R(\pi)$. Contrary to that of the pairing interaction, the sign of the quadrupole-quadrupole interaction $(Q \cdot Q)$ is of no physical importance and is indistinguishable in applications as we have shown above.

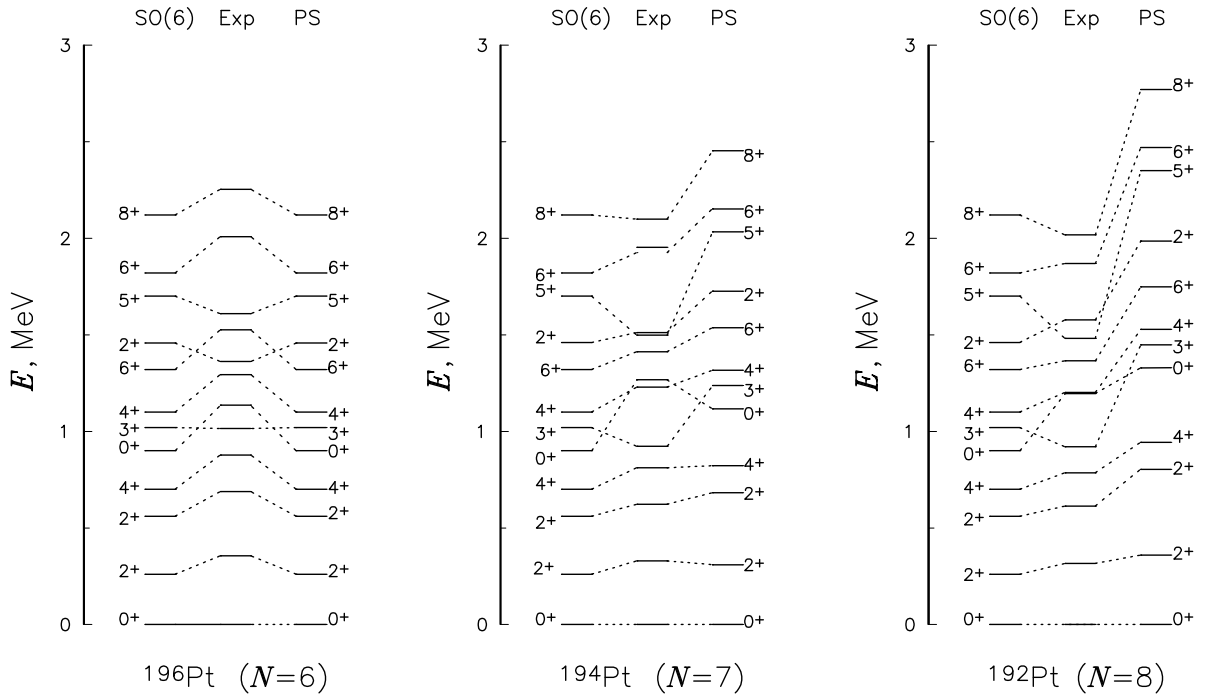

Figure 1: Few lowest levels of each J^π of Pt isotopes. SO(6): calculations within SO(6) DS limit with the parameters suggested in Ref. [4]; PS: calculations with the set of parameters obtained using (6) with $N = 6$; Exp: experimental data of Ref. [12].

3 Parameter Symmetry of IBM-2

IBM-2 [13, 4] is a proton-neutron version of IBM. Within this model, s and d bosons are introduced in the proton and neutron subspaces independently. The symmetry algebra of the model is $\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)$ generated by 72 bilinear operators $s^+_\rho s_\rho$, $d^+_{\rho\mu}d_{\rho\nu}$, $d^+_{\rho\mu}s_\rho$, $s^+_\rho d_{\rho\mu}$ ($\rho = \pi,\, \nu$).

The general IBM-2 Hamiltonian H consists of proton part H_π, neutron part H_ν and proton-neutron interaction $V_{\pi\nu}$,

$$H\left(\{k^\pi_i, k^\nu_i, k_i\}\right) = H_\pi\left(\{k^\pi_i\}\right) + H_\nu\left(\{k^\nu_i\}\right) + V_{\pi\nu}\left(\{k_i\}\right), \tag{18}$$

and is characterized by 21 independent parameters $\{k^\pi_i, k^\nu_i, k_i\}$ [4]. The proton and neutron parts of the Hamiltonian, $H_\rho\left(\{k^\rho_i\}\right)$, $\rho = \pi, \nu$, are just the IBM-1 Hamiltonians and are given by (1) with $H_0 = 0$. It is desirable to express the proton-neutron interaction $V_{\pi\nu}\left(\{k_i\}\right)$ as a superposition of Casimir operators of combined proton-neutron subalgebras $\mathrm{G}_{\pi\nu}$ of the $\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)$ algebra that enter the reduction chains starting with $\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)$ and ending with $\mathrm{SO}_{\pi\nu}(3)$. The generators $\mathcal{G}_{\pi\nu}$ of the combined proton-neutron algebras $\mathrm{G}_{\pi\nu}$ are of the form $\mathcal{G}_{\pi\nu} = \mathcal{G}_\pi + \mathcal{G}_\nu$ where $\mathcal{G}_\pi$ and $\mathcal{G}_\nu$ are generators of the corresponding proton and neutron algebras, respectively. For example, the generators of the $\mathrm{SO}_{\pi\nu}(6)$ algebra are $Q^0_{\pi\mu} + Q^0_{\nu\mu}$, $T^1_{\pi\mu} + T^1_{\nu\mu}$, and $T^3_{\pi\mu} + T^3_{\nu\mu}$.

The $\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)$ algebra has a number of appropriate reduction chains. There are three types of the reduction chains which include U(5), SU(3) and SO(6) subalgebras [5]:

1. U(5) DS chains:

$$\begin{array}{ccccccccc}
\mathrm{U}_\pi(6) & \to & \mathrm{U}_\pi(5) & \to & \mathrm{SO}_\pi(5) & \to & \mathrm{SO}_\pi(3) & & \\
 & \searrow & & \searrow & & \searrow & & \searrow & \\
 & & \mathrm{U}_{\pi\nu}(6) & \to & \mathrm{U}_{\pi\nu}(5) & \to & \mathrm{SO}_{\pi\nu}(5) & \to & \mathrm{SO}_{\pi\nu}(3) \\
 & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\
\mathrm{U}_\nu(6) & \to & \mathrm{U}_\nu(5) & \to & \mathrm{SO}_\nu(5) & \to & \mathrm{SO}_\nu(3) & &
\end{array} \tag{19}$$

2. SU(3) DS chains:

$$\begin{array}{ccccccccc} \mathrm{U}_\pi(6) & \to & \mathrm{SU}_\pi(3) & \to & \mathrm{SO}_\pi(3) & & & & \\ & \searrow & & \searrow & & \searrow & & & \\ & & \mathrm{U}_{\pi\nu}(6) & \to & \mathrm{SU}_{\pi\nu}(3) & \to & \mathrm{SO}_{\pi\nu}(3) & & \\ & \nearrow & & \nearrow & & \nearrow & & & \\ \mathrm{U}_\nu(6) & \to & \mathrm{SU}_\nu(3) & \to & \mathrm{SO}_\nu(3) & & & & \end{array} \tag{20}$$

3. SO(6) DS chains:

$$\begin{array}{ccccccccccc} \mathrm{U}_\pi(6) & \to & \mathrm{SO}_\pi(6) & \to & \mathrm{SO}_\pi(5) & \to & \mathrm{SO}_\pi(3) & & & & \\ & \searrow & & \searrow & & \searrow & & \searrow & & & \\ & & \mathrm{U}_{\pi\nu}(6) & \to & \mathrm{SO}_{\pi\nu}(6) & \to & \mathrm{SO}_{\pi\nu}(5) & \to & \mathrm{SO}_{\pi\nu}(3) & & \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & & & \\ \mathrm{U}_\nu(6) & \to & \mathrm{SO}_\nu(6) & \to & \mathrm{SO}_\nu(5) & \to & \mathrm{SO}_\nu(3) & & & & \end{array} \tag{21}$$

We note that the set of Casimir operators provided by the algebras entering the reduction chains (19)–(21), is not complete enough to express the general IBM-2 Hamiltonian (18). The problem is partly solved by adding the $\overline{\mathrm{SO}}(6)$ DS reduction chains [5] to the reduction chains (19)–(21):

4. $\overline{\mathrm{SO}}(6)$ DS chains:

$$\begin{array}{ccccccccccc} \mathrm{U}_\pi(6) & \to & \overline{\mathrm{SO}}_\pi(6) & \to & \mathrm{SO}_\pi(5) & \to & \mathrm{SO}_\pi(3) & & & & \\ & \searrow & & \searrow & & \searrow & & \searrow & & & \\ & & \mathrm{U}_{\pi\nu}(6) & \to & \overline{\mathrm{SO}}_{\pi\nu}(6) & \to & \mathrm{SO}_{\pi\nu}(5) & \to & \mathrm{SO}_{\pi\nu}(3) & & \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & & & \\ \mathrm{U}_\nu(6) & \to & \overline{\mathrm{SO}}_\nu(6) & \to & \mathrm{SO}_\nu(5) & \to & \mathrm{SO}_\nu(3) & & & & \end{array} \tag{22}$$

The reduction chains (19)–(22) will be referred to as standard DS reduction chains.

Contrary to the Casimir operators $C_2[\overline{\mathrm{SO}}_\pi(6)]$ and $C_2[\overline{\mathrm{SO}}_\nu(6)]$ [see Eq. (12)], the Casimir operator $C_2[\overline{\mathrm{SO}}_{\pi\nu}(6)]$ is an additional independent operator that can be used for the construction of the general IBM-2 Hamiltonian. However we still do not have a complete set of independent Casimir operators. To obtain this set we should look for alternative embeddings of the combined proton-neutron algebras. All the alternative subalgebras can be obtained by applying all possible transformations $R_\rho(\varphi_\rho)$ and $\tilde{R}_\rho(\varphi_\rho)$ to the generators of all subalgebras in the reduction chains (19)–(22). As a result, we obtain alternative subalgebras $\mathrm{G}_\rho^{\alpha_\rho}$ and $\mathrm{G}_{\pi\nu}^{\alpha_\pi\alpha_\nu}$ with generators ${}^{\alpha_\rho}\mathcal{G}_\rho$ and ${}^{\alpha_\pi\alpha_\nu}\mathcal{G}_{\pi\nu} = {}^{\alpha_\pi}\mathcal{G}_\pi + {}^{\alpha_\nu}\mathcal{G}_\nu$, respectively. For example, the generators of the algebra $\mathrm{SO}_{\pi\nu}^{01}(6)$ are $Q_{\pi\mu}^0 - Q_{\nu\mu}^0$, $T_{\pi\mu}^1 + T_{\nu\mu}^1$ and $T_{\pi\mu}^3 + T_{\nu\mu}^3$ [see Eq. (15) for the expression of ${}^1Q_{\nu\mu}^0$]. The $\overline{\mathrm{SO}}_{\pi\nu}(6)$ algebra is the $\mathrm{SO}_{\pi\nu}^{1/2\,1/2}(6)$ algebra in these notations. The $\mathrm{SU}_{\pi\nu}^{0-1}(3)$ algebra is equivalent to the $\mathrm{SU}_{\pi\nu}^*(3)$ algebra introduced in Ref. [9] for the description of triaxial shapes within IBM-2.

In such a way we obtain a large number of alternative algebras. However, some of them are equivalent. For example, any algebra $\mathrm{G}_{\pi\nu}^{\alpha_\pi\alpha_\nu}$ is equivalent to its proton-neutron particle-hole counterpart algebra $\mathrm{G}_{\pi\nu}^{-\alpha_\pi-\alpha_\nu}$ — the relative sign of α_ν and α_π is only important, changing the sign of both α_ν and α_π we do not obtain a new algebra. As follows from our analysis, there exist 2 different realizations of $\mathrm{U}_{\pi\nu}(5)$, 2 different realizations of $\mathrm{SO}_{\pi\nu}(6)$, 2 different realizations of $\overline{\mathrm{SO}}_{\pi\nu}(6)$, and 8 different realizations of $\mathrm{SU}_{\pi\nu}(3)$.

The alternative algebras provide us with Casimir operators that can be used for the construction of $V_{\pi\nu}(\{k_i\})$, however not all of these Casimir operators are independent. For example, the Casimir operators of alternative proton or neutron algebras $\mathrm{G}_\rho^{\alpha_\rho}$ can be expressed through the Casimir operators of untransformed algebras G_ρ [see Eqs. (10) and (12)]; the rank-1 Casimir operator of any of $\mathrm{U}_{\pi\nu}^{\alpha_\pi\alpha_\nu}(5)$ algebras can be expressed through the Casimir operators of $\mathrm{U}_\pi^{\alpha_\pi}(5)$ and $\mathrm{U}_\nu^{\alpha_\nu}(5)$: $C_1[\mathrm{U}_{\pi\nu}^{\alpha_\pi\alpha_\nu}(5)] = C_1[\mathrm{U}_\pi^{\alpha_\pi}(5)] + C_1[\mathrm{U}_\nu^{\alpha_\nu}(5)]$, etc. So, we should choose a set of independent rank-2 Casimir operators of combined proton-neutron subalgebras. We suggest to include in this set the Casimir operators of $\mathrm{U}_{\pi\nu}(6)$, $\mathrm{U}_{\pi\nu}(5)$, $\mathrm{SO}_{\pi\nu}(6)$, $\overline{\mathrm{SO}}_{\pi\nu}(6)$, $\mathrm{SU}_{\pi\nu}(3)$, $\mathrm{SU}_{\pi\nu}^{01}(3)$ and $\mathrm{SU}_{\pi\nu}^{11}(3)$. The Casimir operators of all the rest proton-neutron algebras can be expressed through the ones included in the set, e.g.,

$$\begin{aligned} C_2\left[\mathrm{SU}_{\pi\nu}^{0-1}(3)\right] \;=\; & -C_2\left[\mathrm{SU}_{\pi\nu}^{01}(3)\right] + 2C_2\left[\mathrm{SU}_\pi(3)\right] + 2C_2\left[\mathrm{SU}_\nu^1(3)\right] \\ & + \frac{3}{2}\left\{C_2\left[\mathrm{SO}_{\pi\nu}(3)\right] - C_2\left[\mathrm{SO}_\pi(3)\right] - C_2\left[\mathrm{SO}_\nu(3)\right]\right\}. \end{aligned} \tag{23}$$

The Casimir operator $C_2[\mathrm{U}(6)]$ not defined above can be expressed as

$$C_2\left[\mathrm{U}(6)\right] = (S\cdot S) + \frac{1}{2}(Q^0\cdot Q^0) + \frac{1}{2}(\overline{Q^0}\cdot\overline{Q^0}) + \sum_{\lambda=0}^{4}(T^\lambda\cdot T^\lambda)\ . \tag{24}$$

The proton-neutron interaction $V_{\pi\nu}\left(\{k_i\}\right)$ we express through the set of independent Casimir operators as

$$\begin{aligned} V_{\pi\nu}\left(\{k_i\}\right) \ = \ & H_0 + kC_2[\mathrm{U}_{\pi\nu}(6)] + k_2C_2[\mathrm{U}_{\pi\nu}(5)] + k_3C_2[\mathrm{SO}_{\pi\nu}(5)] \\ & + k_4C_2[\mathrm{SO}_{\pi\nu}(3)] + k_5C_2[\mathrm{SO}_{\pi\nu}(6)] + k_6C_2[\overline{\mathrm{SO}}_{\pi\nu}(6)] \\ & + k_7C_2[\mathrm{SU}_{\pi\nu}(3)] + k_8C_2[\mathrm{SU}^{01}_{\pi\nu}(3)] + k_9C_2[\mathrm{SU}^{11}_{\pi\nu}(3)]\ . \end{aligned} \tag{25}$$

Note that the set of independent Casimir operators is not unique and, as a result, alternative expressions for $V_{\pi\nu}\left(\{k_i\}\right)$ can be suggested. Another possible choice of the operators was used in Ref. [14].

It is seen that the construction of different realizations of boson algebras plays an important role in IBM-2. The incompleteness of boson Hamiltonians in the form of superposition of Casimir invariants of different groups determined by standard DS reduction chains, is a common property of systems of two (or more) independent subsystems, e.g. it is also a property of the vibron model of triatomic molecules with the symmetry algebra $\mathrm{U}_1(4)\otimes\mathrm{U}_2(4)$ [15].

The standard reduction chains (19)–(22) define standard DS limits of IBM-2. As the Casimir operators of $\mathrm{SU}^{01}_{\pi\nu}$ and $\mathrm{SU}^{11}_{\pi\nu}$ are present in the Hamiltonian, we can also define non-standard $\mathrm{SU}^{01}_{\pi\nu}$ and $\mathrm{SU}^{11}_{\pi\nu}$ DS limits of IBM-2 that are associated with non-standard SU^{01} and SU^{11} DS reduction chains, respectively. The definition of non-standard DS limits of the model is, of course, ambiguous because of the ambiguity of definition of the complete set of Casimir operators.

Applying all possible transformations $R_\rho(\varphi_\rho)$ and $\tilde{R}_\rho(\varphi_\rho)$ to all subalgebras in the standard reduction chains (19)–(22), we obtain all alternative reduction chains. Some of these reduction chains appear to be equivalent to some of the others, some of them are equivalent to some of the standard or non-standard DS reduction chains. However the set of independent alternative reduction chains can be easily defined. These independent alternative reduction chains give rise to hidden symmetries of the model. It is interesting that some of the hidden symmetries may be obtained by means of transformations (e.g., by particle-hole transformations) that are not isospectral.

Applying all possible transformations $R_\rho(\varphi_\rho)$ and $\tilde{R}_\rho(\varphi_\rho)$ to the general Hamiltonian (18), we obtain a general IBM-2 Hamiltonian that can be (i) identical to the initial Hamiltonian (18), (ii) non-identical to but isospectral with the initial Hamiltonian (18), or (iii) non-isospectral with the initial Hamiltonian (18). In the case (ii) we obtain standard parameter symmetries of IBM-2 that are valid without restrictions on the parameters of the model. The standard parameter symmetries are listed in Table 2. In the case (iii) we do not immediately obtain parameter symmetries. However, for any possible transformation $R_\rho(\varphi_\rho)$ [or $\tilde{R}_\rho(\varphi_\rho)$] there always can be found some constraints on the parameters $\{k^\pi_i, k^\nu_i, k_i\}$ such that the Hamiltonians $H(\{k^\pi_i, k^\nu_i, k_i\})$ and $R_\rho(\varphi_\rho)\,H(\{k^\pi_i, k^\nu_i, k_i\})$ [or $\tilde{R}_\rho(\varphi_\rho)\,H(\{k^\pi_i, k^\nu_i, k_i\})$] become isospectral even if there is no isospectrality between these Hamiltonians in the general case. Hence in the case (iii) we obtain additional non-standard parameter symmetries that are valid only if the parameters fit some relations. These additional symmetries are listed in Table 3 (the constraining relations for the parameters are given in the first row).

All IBM-2 parameter symmetry relations with the only exception of the parameter symmetry D (see Table 2), involve the total number of proton bosons N_π or/and the total number of neutron bosons N_ν ($N = N_\pi + N_\nu$). Hence there is a principal possibility to distinguish between few parameter sets giving rise to identical spectra by the analysis of the spectra of neighboring isotopes or/and isotones.

4 Summary

We have analyzed canonical transformations of boson operators consistent with the structure of boson Hamiltonian in the cases of IBM-1 and IBM-2, or, equivalently, different realizations of the symmetry algebra of the model and its subalgebras. Analysis of alternatively embedded subalgebras is of a particular

Table 2: Standard parameter symmetry relations for the IBM-2 Hamiltonian

A	B	C	D
$R_\pi(0)\times R_\nu(0)$	$R_\pi(\pi)\times R_\nu(0)$	$R_\pi(0)\times R_\nu(\pi)$	$R_\pi(\pi)\times R_\nu(\pi)$
H_0	$H_0-10(k+2k_6)N$	$H_0-10(k+2k_6)N$	H_0
k_1^π	$k_1^\pi+2k_6^\pi+4(N_\pi+2)(2k_6+k)$	$k_1^\pi+4(N_\pi+2)(2k_6+k)$	$k_1^\pi+2k_6^\pi$
k_2^π	$k_2^\pi+2k_6^\pi-8k_6-4k$	$k_2^\pi-8k_6-4k$	$k_2^\pi+2k_6^\pi$
k_3^π	$k_3^\pi-6k_6^\pi-2k_5+2k_6-8k_7$	$k_3^\pi-2k_5+2k_6-8k_9$	$k_3^\pi-6k_6^\pi-8k_8$
k_4^π	$k_4^\pi+2k_6^\pi$	k_4^π	$k_4^\pi+2k_6^\pi$
k_5^π	$k_5^\pi+4k_6^\pi+2k_5-2k_6+8k_7$	$k_5^\pi+2k_5-2k_6+8k_9$	$k_5^\pi+4k_6^\pi+8k_8$
k_6^π	$-k_6^\pi$	k_6^π	$-k_6^\pi$
k_1^ν	$k_1^\nu+4(N_\nu+2)(k+2k_6)$	$k_1^\nu+2k_6^\nu+4(N_\nu+2)(k+2k_6)$	$k_1^\nu+2k_6^\nu$
k_2^ν	$k_2^\nu-8k_6-4k$	$k_2^\nu+2k_6^\nu-8k_6-4k$	$k_2^\nu+2k_6^\nu$
k_3^ν	$k_3^\nu-2k_5+2k_6-8k_7$	$k_3^\nu-6k_6^\nu-2k_5+2k_6-8k_9$	$k_3^\nu-6k_6^\nu-8k_8$
k_4^ν	k_4^ν	$k_4^\nu+2k_6^\nu$	$k_4^\nu+2k_6^\nu$
k_5^ν	$k_5^\nu+2k_5-2k_6+8k_7$	$k_5^\nu+4k_6^\nu+2k_5-2k_6+8k_9$	$k_5^\nu+4k_6^\nu+8k_8$
k_6^ν	k_6^ν	$-k_6^\nu$	$-k_6^\nu$
k	k	k	k
k_2	k_2	k_2	k_2
k_3	$k_3+2k_5+2k_6+2k+8k_7$	$k_3+2k_5+2k_6+2k+8k_9$	k_3+8k_8
k_4	k_4	k_4	k_4
k_5	$-k_5-k-8k_7$	$-k_5-k-8k_9$	k_5-8k_8
k_6	$-k_6-k$	$-k_6-k$	k_6
k_7	k_7	k_8+k_9	k_8+k_9
k_8	$-k_7+k_9$	k_7-k_9	$-k_8$
k_9	k_7+k_8	k_9	k_7+k_8

importance in the case of IBM-2, because it provides a regular way to construct the general IBM-2 Hamiltonian.

One can suppose that there should be no signals in physical applications of switching from one realization of the symmetry algebra to another equivalent realization. However, it is not so. The existence of alternative realizations of the symmetry algebra manifests itself as parameter symmetries of the model, i.e. as existence of few sets of Hamiltonian parameters providing identical spectra. The parameter symmetry is of physical importance for applications of IBM since the parameters of the model are obtained by the fit to experimental spectra. We have shown that in some cases one can discriminate between the sets of parameters related by the parameter symmetry by analyzing spectra of neighboring isotopes and/or isotones.

The parameter symmetry is a common property of boson models. For example, we have shown [16] that it is present in the vibron model (see e.g. [5]); we suppose that it can be found in *sdg*-IBM and other algebraic models including fermion and boson-fermion ones. There can exist other possibilities of discriminating between isospectral parameter sets (see, e.g., Ref. [17] where the proton-neutron interacting boson-fermion model is discussed).

Before finishing the paper, we mention few recent papers related to the present investigation. D. Kusnezov [6] discussed in detail hidden symmetries, i.e. the particular cases of parameter symmetries relating transitional Hamiltonians to the ones corresponding to DS limits. He noted the relevance of hidden symmetries to the studies of chaos. A more detailed discussion of this item can be found in Ref. [18]. In Ref. [19] the so-called d parity was introduced for IBM Hamiltonian in the U(5)–SO(6) transitional case. The d parity operator commutes with the Hamiltonian and provides an additional quantum number for qualification of the energy levels, electromagnetic transitions, etc.

Table 3: Parameter symmetry relations for the IBM-2 Hamiltonian with some constraints on the parameters

A	E	F	G	H	I
	$k+2k_5+8k_7=0$ $k+2k_6=0$ $k_9=k_7+k_8$ $k_6^\pi=0$	$k+2k_5+8k_9=0$ $k+2k_6=0$ $k_7=k_8+k_9$ $k_6^\nu=0$	$k_8=0$ $k_7=k_9$ $k_6^\pi=0$ $k_6^\nu=0$	$k+2k_5+8k_7=0$ $k+2k_6=0$ $k_9=k_7+k_8$ $k_6^\pi=0$	$k+2k_5+8k_9=0$ $k+2k_6=0$ $k_7=k_8+k_9$ $k_6^\nu=0$
	$R_\pi(\pi/2)\times R_\nu(0)$	$R_\pi(0)\times R_\nu(\pi/2)$	$R_\pi(\pi/2)\times R_\nu(\pi/2)$	$R_\pi(\pi/2)\times R_\nu(\pi)$	$R_\pi(\pi)\times R_\nu(\pi/2)$
H_0	$H_0-10(k_5^\pi+4k_8)N_\pi$	$H_0-10(k_5^\nu+4k_8)N_\nu$	$H_0-10(k_5^\pi N_\pi-k_5^\nu N_\nu)$	$H_0-10(k_5^\pi+4k_8)N_\pi$	$H_0-10(k_5^\nu+4k_8)N_\nu$
k_1^π	$k_1^\pi+4(N_\pi+2)(k_5^\pi+4k_8)$	k_1^π	$k_1^\pi+4(N_\pi+2)k_5^\pi$	$k_1^\pi+4(N_\pi+2)(k_5^\pi+4k_8)$	$k_1^\pi+2k_7^\pi$
k_2^π	$k_2^\pi-4k_5^\pi-16k_8$	k_2^π	$k_2^\pi-4k_5^\pi$	$k_2^\pi-4k_5^\pi-16\tilde{k}_6$	$k_2^\pi+2k_6^\pi$
k_3^π	$k_3^\pi+2k_5^\pi+8k_8$	k_3^π	$k_3^\pi+2k_5^\pi$	$k_3^\pi+2k_5^\pi$	$k_3^\pi-6k_6^\pi-8k_8$
k_4^π	k_4^π	k_4^π	k_4^π	k_4^π	$k_4^\pi+2k_6^\pi$
k_5^π	$-k_5^\pi-8k_8$	k_5^π	$-k_5^\pi$	$-k_5^\pi$	$k_5^\pi+4k_6^\pi+8k_8$
k_6^π	$k_6^\pi=0$	k_6^π	$k_6^\pi=0$	$k_6^\pi=0$	$-k_6^\pi$
k_1^ν	k_1^ν	$k_1^\nu+4(N_\nu+2)(k_5^\nu+4k_8)$	$k_1^\nu+4(N_\nu+2)k_5^\nu$	$k_1^\nu+2k_6^\nu$	$k_1^\nu+4(N_\nu+2)(k_5^\nu+4k_8)$
k_2^ν	k_2^ν	$k_2^\nu-4k_5^\nu-16k_8)$	$k_2^\nu-4k_5^\nu$	$k_2^\nu+2k_6^\nu$	$k_2^\nu-4k_5^\nu-16k_8$
k_3^ν	k_3^ν	$k_3^\nu+2k_5^\nu+8k_8$	$k_3^\nu+2k_5^\nu$	$k_3^\nu-6k_6^\nu-8k_8$	$k_3^\nu+2k_5^\nu$
k_4^ν	k_4^ν	k_4^ν	k_4^ν	$k_4^\nu+2k_6^\nu$	k_4^ν
k_5^ν	k_5^ν	$-k_5^\nu-8k_8$	$-k_5^\nu$	$k_5^\nu+4k_6^\nu+8k_8$	$-k_5^\nu$
k_6^ν	k_6^ν	$k_6^\nu=0$	$k_6^\nu=0$	$-k_6^\nu$	$k_6^\nu=0$
k	k	k	k	k	k
k_2	k_2	k_2	k_2	k_2	k_2
k_3	k_3	k_3	k_3	k_3+8k_8	k_3+8k_8
k_4	k_4	k_4	k_4	k_4	k_4
k_5	k_5	k_5	k_6-4k_7	k_5-8k_8	k_5-8k_8
k_6	k_6	k_6	k_5+4k_7	k_6	k_6
k_7	k_7	k_7	k_7	k_8+k_9	k_7
k_8	k_8	k_8	$k_8=0$	$-k_8$	$-k_8$
k_9	k_9	k_9	k_9	k_9	k_7+k_8

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